

Three-body correlations and finite-size effects in the Moore–Read states on a sphere

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Two- and three-body correlations in partially filled degenerate fermion shells are studied numerically for various interactions between the particles. Three distinct correlation regimes are defined, depending on the short-range behavior of the pair pseudopotential. For pseudopotentials similar to those of electrons in the first excited Landau level, correlations at half-filling have a simple three-body form consisting of the maximum avoidance of the triplet state with the smallest relative angular momentum $\mathcal{R}_3 = 3$. In analogy to the superharmonic criterion for Laughlin two-body correlations, their occurrence is related to the form of the three-body pseudopotential at short range. The spectra of a model three-body repulsion are calculated, and the zero-energy Moore–Read ground state, its $(\pm e/4)$ -charged quasiparticles, and the magnetoroton and pair-breaking bands are all identified. The quasiparticles are correctly described by a composite fermion model appropriate for Halperin’s p -type pairing with Laughlin correlations between the pairs. However, the Moore–Read ground state, and specially its excitations, have small overlaps with the corresponding Coulomb eigenstates when calculated on a sphere. The reason lies in surface curvature which affects the form of pair pseudopotential for which the “ $\mathcal{R}_3 > 3$ ” three-body correlations occur. In finite systems, such pseudopotential must be slightly superharmonic at short range (different from Coulomb pseudopotential). However, the connection with the three-body pseudopotential is less size-dependent, suggesting that the Moore–Read state and its excitations are a more accurate description for experimental $\nu = \frac{5}{2}$ states than could be expected from previous calculations.

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I. INTRODUCTION

The fractional quantum Hall (FQH) effect^{1,2} is a many-body phenomenon consisting of the quantization of Hall conductance and the simultaneous vanishing of longitudinal resistance of a high-mobility quasi-two-dimensional electron gas at a strong magnetic field B and low density ρ , corresponding to certain universal fractional values of the Landau level (LL) filling factor $\nu = 2\pi\rho\lambda^2$ (where $\lambda = \sqrt{\hbar c/eB}$ is the magnetic length). This macroscopic phenomenon is a consequence of the formation of incompressible liquid ground states (GS’s) with quasiparticle (QP) excitations.² It depends on correlations in partially filled degenerate LL’s, entirely determined by a Haldane pseudopotential³ defined as the pair interaction energy V_2 as function of relative pair angular momentum \mathcal{R}_2 .

The Haldane hierarchy^{4,5,6,7,8} of most prominent FQH states in LL_0 (lowest LL), equivalent to Jain’s sequence⁹ of filled composite fermion^{10,11} (CF) levels, results for Laughlin correlations^{12,13,14} (between electrons or QP’s) induced by pseudopotentials strongly superharmonic^{15,16} at short range. However, the FQH states with different, non-Laughlin correlations occur as well. E.g., pairing in a half-filled LL_1 (first excited LL) is firmly established in the $\nu = \frac{5}{2}$ state,^{17,18,19,20} while correlations between CF’s in their CF- LL_1 responsible for the FQH effect^{21,22} at $\nu = \frac{3}{8}$ or $\frac{4}{11}$ are not yet completely understood.

The lack of superharmonic behavior of the pseudopotential at short range together with the occurrence of clearly non-Laughlin half-filled FQH states suggests pairing in both LL_1 and CF- LL_1 . Proposed trial states include Halperin¹² and Haldane–Rezayi²³ states with

Laughlin correlations between spin-triplet and -singlet pairs, respectively, and the Moore–Read^{24,25} pfaffian state that can be defined as a zero-energy ground state of a short-range three-body repulsion.²⁶ These pair states have all been studied in great detail^{27,28,29,30,31,32,33} because of their anticipated exotic properties, such as non-abelian QP statistics²⁴ or existence of pair-breaking neutral fermion excitations.²⁶ However, choosing the correct one for specific real FQH systems is somewhat problematic. In the following we concentrate on the half-filled LL_1 . The question of pairing in CF- LL_1 is addressed elsewhere.³⁴

The trouble with the Halperin state¹² is that because the relative angular momentum of the constituent pairs is not a conserved quantity, it is more of an intuitive concept for the correlations than a well-defined trial wavefunction obeying all required symmetries. E.g., description of the pair–pair interaction by an effective pseudopotential is not rigorous,³⁴ and the harmonic criterion^{15,16} that would relate the occurrence of Laughlin pair–pair correlations with the electron pseudopotential is not exact. Consequently, it has not been clear what exactly is the model interaction that induces such correlations (and such ground state). In fact, it has been (erroneously) assumed²⁶ that this paired state results for pseudopotentials attractive at short range rather than harmonically repulsive as in LL_1 , which would suggest that it is not an adequate trial state for the $\nu = \frac{5}{2}$ FQH effect.

The Moore–Read wavefunction on the other hand is well-defined.^{24,25,26} However, it only occurs for interactions with very particular short-range behavior, while the pseudopotentials in realistic experimental systems depend on sample parameters like the layer width w , magni-

tude and tilt of the magnetic field, etc. Moreover, finite-size calculations indicate that realistic Coulomb pseudopotentials are too weak at short range (by up to $\sim 10\%$ for $w = 0$) to induce a Moore–Read ground state.^{25,35} This would seem to imply that the Moore–Read state does not describe the $\nu = \frac{5}{2}$ FQH state quite as accurately as a Laughlin state describes the actual $\nu = \frac{1}{3}$ ground states. The occurrence of the $\nu = \frac{5}{2}$ FQH effect could still be attributed to the observation that the calculated excitation gaps are much less sensitive to the details of the pseudopotential than the wavefunctions. However, poor accuracy of the Moore–Read wavefunction puts doubt on the occurrence of those of its properties in realistic $\nu = \frac{5}{2}$ systems that depend more critically on the correlations. As these properties (including non-abelian QP’s) are so much more fascinating than plain incompressibility, the question of whether they indeed remain only an unrealized theoretical concept is quite significant. Theoretical insight is especially valuable in this problem because of the difficulty with direct experimental evidence.³⁶

In this paper we report on numerical calculations of three-body correlation functions (defined in analogy to Haldane pair amplitudes³) of the half-filled shells with model pair interactions. We find that the vanishing of the triplet amplitude $\mathcal{G}_3(\mathcal{R}_3)$ for the minimum triplet relative angular momentum $\mathcal{R}_3 = 3$, distinctive for the Moore–Read state, occurs for the slightly superharmonic pseudopotential, different from the nearly harmonic one of LL_1 . When the vanishing of $\mathcal{G}_3(3)$ is related to the triplet rather than pair pseudopotential, it becomes evident that the short-range anharmonicity of the critical pair interaction is a finite-size curvature effect on a sphere that could disappear in the thermodynamical limit. Consequently, the three-body correlations defining the Moore–Read state and consisting of the avoidance of the $\mathcal{R} = 3$ hard-core appear to be a much better description of the real $\nu = \frac{5}{2}$ electron systems than expected before.

Based on the anharmonicity of the triplet pseudopotential we also argue that the Halperin paired state is not an adequate model for subharmonic interactions (e.g., in CF-LL_1) because of the tendency to form larger clusters. However, the avoidance of $\mathcal{R}_3 = 3$ triplet states coinciding with an increased number of $\mathcal{R}_2 = 1$ pairs (compared to the minimum value of a Laughlin-correlated state) that occurs for the harmonic interactions is precisely the signature of Halperin’s pairing concept, which therefore is established as a valid model for the Moore–Read state and its excitations. Indeed, the energy spectra of the model three-body repulsion show the low-energy bands containing quasielectron (QE) and quasihole (QH) excitations of charge $\mathcal{Q} = \pm e/4$, in perfect agreement with Halperin’s picture for the Laughlin state of $\mathcal{R}_2 = 1$ pairs (with the obvious exception being the additional pair-breaking excitation²⁶).

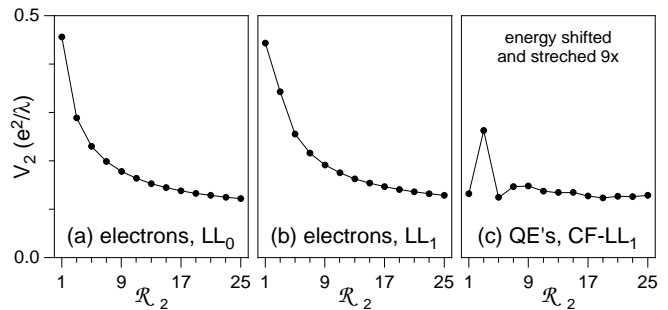


FIG. 1: Pair interaction pseudopotentials (pair interaction energy V_2 vs. relative pair angular momentum \mathcal{R}_2) for electrons in the lowest (a) and first excited LL (b), and for QE’s of the Laughlin $\nu = \frac{1}{3}$ state (c). The values of V_2 in frame (c) were calculated by Lee *et al.*³⁷ and are only known up to a constant. λ is the magnetic length.

II. TWO-BODY CORRELATIONS

A. Haldane pair pseudopotential

Within a degenerate LL, the many-body Hamiltonian only contains the interaction term, which is completely determined by the discrete (Haldane) pseudopotential $V_2(\mathcal{R}_2)$ defined as pair interaction energy V_2 as a function of relative pair angular momentum \mathcal{R}_2 . For identical fermions/bosons, \mathcal{R}_2 takes on odd/even integer values, respectively, and the larger \mathcal{R}_2 corresponds to a larger average pair separation $\sqrt{\langle r^2 \rangle}$. On a sphere, $\mathcal{R}_2 = 2l - L_2$ where l is the single-particle angular momentum of the shell (LL), and L_2 is the total pair angular momentum. (We use the following standard notation for Haldane⁴ spherical geometry: $l = Q + n$ for the n th LL, $2Q = 4\pi R^2 B / \phi_0$ is the magnetic monopole strength $\phi_0 = hc/e$ is the flux quantum, R is the sphere radius, and $\lambda = R/\sqrt{Q}$ is the magnetic length.) Importantly, $V_2(\mathcal{R}_2)$ combines information about both interaction potential $V(r)$ and the single-particle wavefunctions allowed within the Hilbert space restricted to a LL. The pseudopotentials obtained for the electrons in LL_0 and LL_1 , and for Laughlin QE’s in CF-LL_1 are shown in Fig. 1.

B. Pair amplitudes

The pair correlations induced by a specific $V_2(\mathcal{R}_2)$ are conveniently described by a discrete pair amplitude function $\mathcal{G}_2(\mathcal{R}_2)$, defined as the number of pairs \mathcal{N}_2 with a given \mathcal{R}_2 divided by the total pair number,

$$\mathcal{G}_2(\mathcal{R}_2) = \binom{N}{2}^{-1} \mathcal{N}_2(\mathcal{R}_2). \quad (1)$$

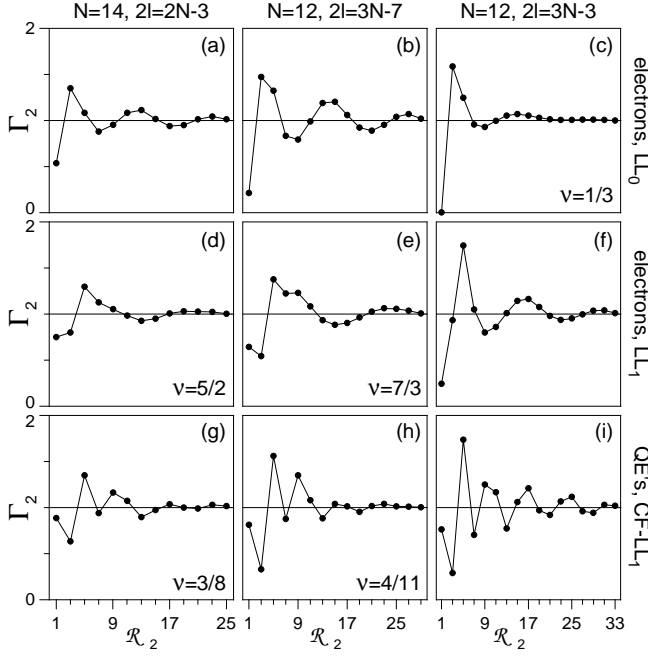


FIG. 2: Pair-correlation functions (pair amplitude Γ_2 vs. relative pair angular momentum \mathcal{R}_2) calculated on a sphere for the lowest $L = 0$ states of N particles interacting through pseudopotentials shown in Fig. 1, at values of $2l$ corresponding to different FQH states at filling factors $\nu = \frac{1}{2}$ and $\frac{1}{3}$.

It immediately follows from the expression for the total interaction energy of an N -body state,

$$E = \binom{N}{2} \sum_{\mathcal{R}_2} \mathcal{G}_2(\mathcal{R}_2) V_2(\mathcal{R}_2), \quad (2)$$

that the low-energy many-body states generally have a large/small amplitude at those values of \mathcal{R}_2 corresponding to small/large repulsion $V_2(\mathcal{R}_2)$. In Fig. 2 we compare the pair amplitudes obtained in Haldane spherical geometry for $N = 12$ and 14 particles confined in angular momentum shells with degeneracy $g = 2l + 1$ corresponding to the filling factors $\nu \sim \frac{1}{3}$ and $\frac{1}{2}$ and interacting through the pseudopotentials of Fig. 1. Although for each system (V_2, N, g) we only show the data for the lowest $L = 0$ state, virtually identical $\mathcal{G}_2(\mathcal{R}_2)$ functions are obtained for all low-energy states of each system.

The chosen values of $2l$ and N correspond to three different sequences of finite-size spherical systems known to represent the following FQH states observed experimentally on a plane. The $2l = 2N - 3$ sequence describes the paired Moore–Read²⁴ $\nu = \frac{1}{2}$ state in LL_1 (corresponding to the total electron filling factor $\nu = \frac{5}{2}$) and the $\nu = \frac{1}{2}$ state of QE’s in CF- LL_1 identified numerically³⁴ for $N = 6, 10$, and 14 , and corresponding to the FQH effect²¹ at $\nu = \frac{3}{8}$. The $2l = 3N - 7$ sequence describes the (not well understood) $\nu = \frac{1}{3}$ states in both LL_1 ¹⁶ and CF- LL_1 ,³⁴ corresponding to the $\nu = \frac{7}{3}$ state^{17,18,19,20} and $\nu = \frac{4}{11}$ state,^{21,22} respectively. Finally, the $2l = 3N - 3$

sequence describes the Laughlin² $\nu = \frac{1}{3}$ state in LL_0 .

The pair amplitude calculated for a completely filled shell (the $\nu = 1$ state) with a given $2l$ is a decreasing straight line,

$$\mathcal{G}_2^{\text{full}}(\mathcal{R}_2) = \frac{4l + 1 - 2\mathcal{R}_2}{l(2l + 1)}, \quad (3)$$

which is a finite-size edge effect. In the $2l \rightarrow \infty$ limit corresponding to an infinite plane, $\mathcal{N}_2^{\text{full}}(\mathcal{R}_2) = N$ and the ratio $\mathcal{N}_2^{\text{full}}(\mathcal{R}_2)/N \equiv \Gamma_2^{\text{full}}(\mathcal{R}_2) = 1$ is the appropriately renormalized pair amplitude in this geometry.

The overall linear decrease of $\mathcal{G}_2(\mathcal{R}_2)$ appears also at $\nu < 1$, and it should be ignored in the analysis of correlations. Therefore, in Fig. 2 we actually plot

$$\Gamma_2(\mathcal{R}_2) = 1 + \frac{\mathcal{G}_2(\mathcal{R}_2) - \mathcal{G}_2^{\text{full}}(\mathcal{R}_2)}{\mathcal{G}_2^{\text{full}}(1)}, \quad (4)$$

in which the linear decrease is eliminated and the scaling appropriate for an infinite plane is used, to ensure that $\Gamma_2(1) \propto \mathcal{G}_2(1)$, that $\Gamma_2(\mathcal{R}_2) = 1$ for finite-size $\nu = 1$ states, and that $\Gamma_2(\mathcal{R}_2)$ converges to the pair-correlation function on the plane when N is increased.

In all frames, Γ_2 is significantly different from 1 only at small \mathcal{R}_2 , and the oscillations around this value quickly decay beyond $\mathcal{R}_2 \sim 7$. This can be interpreted as a short correlation range ξ in all studied systems and it justifies the use of finite-size calculations (requiring that $\xi \ll R$).

Clearly, three different interactions result in quite different correlations. In LL_0 (a–c), the dominant tendency is the avoidance of $\mathcal{R}_2 = 1$ (Laughlin correlations) at a cost of having a large number of pairs with $\mathcal{R}_2 = 3$. Around half-filling of LL_1 (d,e), the numbers of pairs with $\mathcal{R}_2 = 1$ and 3 are about equal and both small. Finally, in a partially filled CF- LL_1 (g–i), the $\mathcal{R}_2 = 3$ pair state is maximally avoided.

C. Model interaction and pair-correlation regimes

The fact that $\Gamma_2 \approx 1$ at long range explains also why the low-energy wavefunctions are virtually insensitive to the exact form of $V_2(\mathcal{R}_2)$ beyond a few leading parameters at $\mathcal{R}_2 = 1, 3, \dots$. Furthermore, due to the sum rules obeyed by pair amplitudes,^{15,16,38} the harmonic pseudopotentials $V_2^H(\mathcal{R}_2) = c_0 - c_1\mathcal{R}$ (with constant c_0 and c_1) induce no correlations, and only the anharmonic contributions to $V_2(\mathcal{R}_2)$ at small \mathcal{R}_2 (short range) affect the pair-correlation functions. Indeed, simple model pseudopotentials with only two nonvanishing leading parameters are known^{16,26,34} to accurately reproduce correlations shown in Fig. 2. Let us define such $U_\alpha(\mathcal{R}_2)$ with

$$\begin{aligned} U_\alpha(1) &= 1 - \alpha, \\ U_\alpha(3) &= \alpha/2. \end{aligned} \quad (5)$$

U_0 and U_1 are the two extremal pseudopotentials with only one anharmonic term. $U_{\frac{1}{2}}$ is harmonic through

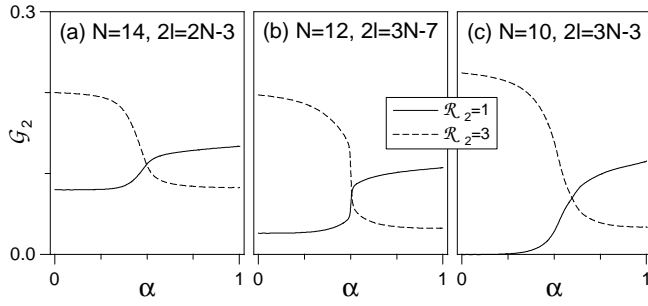


FIG. 3: Dependence of pair amplitudes \mathcal{G}_2 on parameter α of pair interaction U_α defined by Eq. (5), calculated on a sphere for the lowest $L = 0$ states of N -particle systems representing the same FQH states as used in Fig. 2.

$\mathcal{R}_2 = 1, 3$, and 5 , and thus it favors equally the avoidance of both $\mathcal{R}_2 = 1$ and 3 pairs (any distribution of the pair amplitude between the $\mathcal{R}_2 = 1, 3$, and 5 states that satisfies the sum rules yields the same total energy E).

In Fig. 3 we plot $\mathcal{G}_2(1)$ and $\mathcal{G}_2(3)$ as a function of α for the lowest $L = 0$ state in three finite-size systems representing the same series of FQH states as used in Fig. 2. The correlations in a partially filled LL_0 (Laughlin correlations), LL_1 , and CF-LL_1 are well reproduced by U_α with $\alpha \approx 0, \frac{1}{2}$, and 1 , respectively. The correlations at $\alpha = 0$ (i.e., in LL_0) and at $\alpha = 1$ (i.e., in CF-LL_1) can easily be expressed in terms of pair amplitudes. With $U(1)$ or $U(3)$ being the only nonvanishing (and positive) coefficient, it follows from Eq. (2) that the low-energy states must have the minimum allowed (within the available Hilbert space) $\mathcal{G}_2(1)$ or $\mathcal{G}_2(3)$, respectively.

In case of Laughlin correlations, because of the simple form of single-particle wavefunctions in LL_0 , the complete avoidance of the $\mathcal{R}_2 = 1$ pairs (possible at $\nu \leq \frac{1}{3}$) appears in form of a Jastrow factor in the Laughlin wavefunction. It justifies the mean-field CF picture that essentially attributes the reduction of the many-body degeneracy caused by the $\mathcal{R}_2 = 1$ hard-core (or “correlation hole”) to an effective, reduced magnetic field.

For QE’s, the tendency to have small $\mathcal{G}_2(3)$ and, consequently, significant $\mathcal{G}_2(1)$ (compared to a Laughlin-correlated state at the same ν) has been interpreted as $\mathcal{R}_2 = 1$ pairing (although most recent numerical studies are not conclusive about how the pairs correlate with one another, and thus the question of the origin of the excitation gap observed at $\nu = \frac{3}{8}$ or $\frac{4}{11}$ remains open).

In a partially filled LL_1 the situation is more complicated. Because $V_2(\mathcal{R}_2)$ is nearly harmonic at short range ($\alpha \sim \frac{1}{2}$), the energy is nearly independent of the relative occupation of the $\mathcal{R}_2 = 1$ and 3 pair states. Therefore, the correlations cannot be easily expressed in terms of pair amplitudes (although the linear combination of $\mathcal{G}_2(1)$ and $\mathcal{G}_2(3)$ equal to the total energy E is obviously minimized at its corresponding value of α). However, it turns out that it is the short-range three-body correlations that determine the low-energy states in this regime. Soon after its introduction, the

half-filled Moore–Read state was shown²⁶ to be an exact zero-energy eigenstate of a model short-range three-body repulsion, and the spectra of this interaction were later studied in detail.^{27,30} Below we analyze the three-body correlations directly, by the calculation of an appropriate correlation function.

III. THREE-BODY CORRELATIONS

A. Three-body pseudopotential

In analogy to the avoidance of the strongly repulsive pair states, the three-body states with sufficiently high energy (compared to the rest of the three-body spectrum) will also be avoided in the low-energy many-body states. For the pairs, the eigenstates are uniquely labeled by \mathcal{R}_2 , and the criterion for the avoidance of a specific \mathcal{R}_2 is^{15,16} that it corresponds to the dominant positive anharmonic term of $V_2(\mathcal{R}_2)$. The three-body states are also labeled by the relative (with respect to the center of mass) angular momentum \mathcal{R}_3 . The allowed values are $\mathcal{R}_3 = 3$ or $\mathcal{R}_3 \geq 5$, and larger \mathcal{R}_3 means larger expectation value of the area spanned by the three particles.³⁹ On a sphere, $\mathcal{R}_3 = 3l - L_3$, where L_3 is the total triplet angular momentum.

Since no degeneracies appear in the $V_3(\mathcal{R}_3)$ energy spectrum for $\mathcal{R}_3 < 9$, its low- \mathcal{R}_3 part can be considered a three-body pseudopotential analogous to $V_2(\mathcal{R}_2)$. The three-body pseudopotentials $V_3(\mathcal{R}_3)$ obtained for different pair pseudopotentials $V_2(\mathcal{R}_2)$ of Fig. 1 are shown in the upper frames of Fig. 4. The nonmonotonic behavior of $V_3(\mathcal{R}_3)$ in frame (c) most likely precludes the tendency to avoid the $\mathcal{R}_3 = 3$ triplet state in QE systems. On the other hand, it seems plausible that the monotonic character of $V_3(\mathcal{R}_3)$ in frame (b) might lead to the avoidance of the same $\mathcal{R}_3 = 3$ triplet state in a partially filled LL_1 .

The dependence of $V_3(\mathcal{R}_3)$ on $V_2(\mathcal{R}_2)$ can be captured by plotting the leading V_3 coefficients as a function of parameter α of the model pair pseudopotential U_α , as shown in frame (d). For $\mathcal{R}_3 < 9$ the triplet wavefunctions are fixed and so are their \mathcal{G}_2 amplitudes, and hence the dependences $V_3(\alpha)$ are all linear. Only around $\alpha \sim \frac{1}{2}$ is the $V_3(\mathcal{R}_3)$ function superlinear for small \mathcal{R}_3 , as shown on an example for $\alpha = 0.54$ in frame (e).

B. Three-body amplitudes

In order to test the hypothesis of the avoidance of the $\mathcal{R}_3 = 3$ triplet eigenstate in partially filled LL_1 , we introduce “triplet amplitude” $\mathcal{G}_3(\mathcal{R}_3)$. It is defined in analogy to the pair amplitude, as an expectation value of the operator $\hat{\mathcal{P}}_{ijk}(\mathcal{R}_3, \beta_3)$ projecting a many-body state Ψ onto the subspace in which the three particles ijk are in an eigenstate $|\mathcal{R}_3, \beta_3\rangle$ (here, β_3 is an additional index to distinguish degenerate multiplets at the same \mathcal{R}_3 ; it can be omitted for $\mathcal{R}_3 < 9$). The interaction Hamiltonian

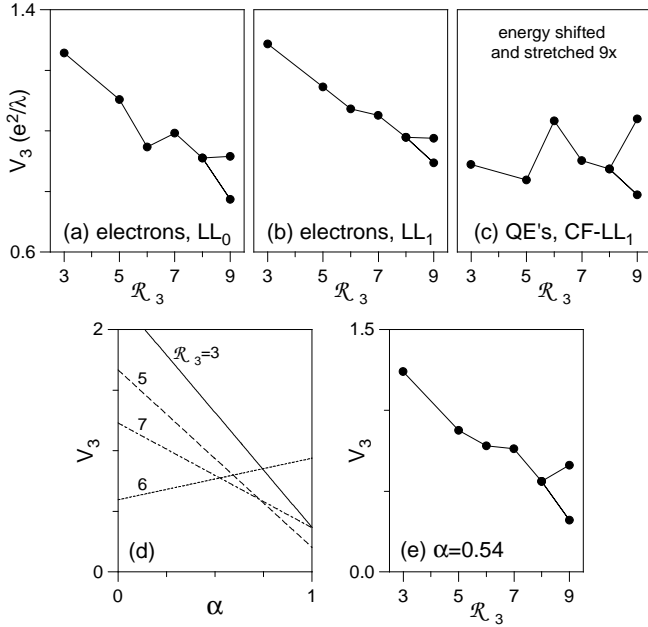


FIG. 4: (a,b,c) Triplet interaction pseudopotentials (triplet interaction energy V_3 vs. relative triplet angular momentum \mathcal{R}_3) for pair pseudopotentials shown in Fig. 1. λ is the magnetic length. (d) Dependence of coefficients V_3 on parameter α of pair interaction U_α defined by Eq. (5). (e) Pseudopotential $V_3(\mathcal{R}_3)$ calculated for pair interaction $U_{0.54}$.

written in a three-body form using $\hat{\mathcal{P}}_{ijk}$ reads

$$\hat{\mathcal{H}} = \sum_{i < j < k} \sum_{\mathcal{R}_3, \beta_3} \hat{\mathcal{P}}_{ijk}(\mathcal{R}_3, \beta_3) V(\mathcal{R}_3, \beta_3). \quad (6)$$

The triplet amplitude is

$$\mathcal{G}(\mathcal{R}_3, \beta_3) = \binom{N}{3}^{-1} \langle \Psi | \sum_{i < j < k} \hat{\mathcal{P}}_{ijk}(\mathcal{R}_3, \beta_3) | \Psi \rangle, \quad (7)$$

which for a totally antisymmetric Ψ is equivalent to

$$\mathcal{G}(\mathcal{R}_3, \beta_3) = \langle \Psi | \hat{\mathcal{P}}_{123}(\mathcal{R}_3, \beta_3) | \Psi \rangle. \quad (8)$$

Pair amplitudes defined in this way are normalized to

$$\sum_{\mathcal{R}_3, \beta_3} \mathcal{G}_3(\mathcal{R}_3, \beta_3) = 1, \quad (9)$$

so that they measure the fraction of all triplets being in a given eigenstate,

$$\mathcal{G}_3(\mathcal{R}_3, \beta_3) = \binom{N}{3}^{-1} \mathcal{N}_3(\mathcal{R}_3, \beta_3). \quad (10)$$

The energy of Ψ is expressed as

$$E = \binom{N}{3} \sum_{\mathcal{R}_3, \beta_3} \mathcal{G}_3(\mathcal{R}_3, \beta_3) V_3(\mathcal{R}_3, \beta_3). \quad (11)$$

On a sphere, triplet amplitudes are connected with the third-order parentage coefficients⁴⁰ $G_3(L_3, \beta_3; L'_3, \beta'_3)$, i.e., the expansion coefficients of a totally antisymmetric state Ψ in a basis of product states in which particles (1, 2, 3) and (4, 5, ..., N) are in the 3- and ($N-3$)-body eigenstates $|L_3, \beta_3\rangle$ and $|L'_3, \beta'_3\rangle$, respectively,

$$\mathcal{G}_3(L_3, \beta_3) = \sum_{L'_3, \beta'_3} |G_3(L_3, \beta_3; L'_3, \beta'_3)|^2. \quad (12)$$

Note that to obey standard notation for parentage coefficients, in the above equation we use total angular momentum L_3 instead of the relative one $\mathcal{R}_3 = 3l - L_3$ to label triplet states. Also, we omit index Ψ in \mathcal{G}_3 and G_3 .

The following operator identity³⁸

$$\hat{L}^2 + N(N-2)\hat{l}^2 = \sum_{i < j} \hat{L}_{ij}^2 \quad (13)$$

connects the total N -body angular momentum (L) with the single-particle and pair angular momenta l and L_{ij} . We used it earlier to show that harmonic pair pseudopotentials cause no correlations. It can be generalized to the following form

$$\hat{L}^2 + \frac{N(N-K)}{K-1} \hat{l}^2 = \binom{N-2}{K-2}^{-1} \sum_{i_1 < \dots < i_K} \hat{L}_{i_1 \dots i_K}^2, \quad (14)$$

By taking the expectation values of both sides of the above equation in the (totally antisymmetric) state Ψ and using the expansion of Ψ in terms of the K th-order parentage coefficients we obtain

$$\begin{aligned} L(L+1) + \frac{N(N-K)}{K-1} l(l+1) \\ = \frac{N(N-1)}{K(K-1)} \sum_{L_K, \beta_K} \mathcal{G}_K(L_K, \beta_K) L_K(L_K+1), \end{aligned} \quad (15)$$

an additional (besides normalization) sum rule obeyed by the amplitudes \mathcal{G}_K .

Just as for the specific $K=2$ case discussed earlier,³⁸ the above sum rule (15) together with an appropriate version of Eq. (2) or (11) immediately implies that if the K -body interaction pseudopotential V_K is linear in $L_K(L_K+1)$, all N -body multiplets with the same L are degenerate. In the limit of infinite LL degeneracy $g = 2l + 1$ corresponding to an infinite sphere radius (vanishing curvature) i.e., to the planar geometry, the linearity in $L_K(L_K+1)$ translates into the linearity in $\mathcal{R}_K = Kl - L_K$, and it turns out that the linear part of $V_K(\mathcal{R}_K)$ causes no correlations.

C. Three-body correlation hole

Let us now turn back to the numerical results. In Fig. 5 we plot the dependence of the leading $\mathcal{G}_3(\mathcal{R}_3)$ coefficients

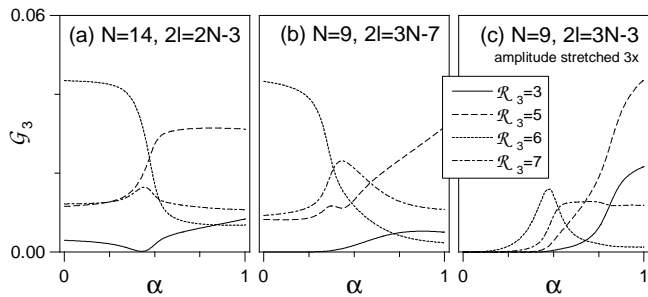


FIG. 5: Dependence of triplet amplitudes \mathcal{G}_3 on parameter α of pair interaction U_α defined by Eq. (5), calculated on a sphere for the lowest $L = 0$ states of N -particle systems representing the same FQH states as used in Figs. 2 and 3.

on α , calculated in the lowest $L = 0$ state of three different systems belonging to the same sequences of finite-size FQH states as used earlier in Figs. 2 and 3. Clearly, all triplet amplitudes significantly depend on α , but we especially want to point out the following three features for $\mathcal{R}_3 = 3$: (i) the tendency to avoid $\mathcal{R}_2 = 1$ pairs at $\alpha \sim 0$ is *not* synonymous with the avoidance of $\mathcal{R}_3 = 3$ triplets at $\nu = \frac{1}{2}$, (ii) $\mathcal{G}_3(3)$ *vanishes* for $\alpha \approx \frac{1}{2}$ at $\nu = \frac{1}{2}$. (iii) $\mathcal{G}_3(3)$ *increases* when α increases beyond $\frac{1}{2}$ in all frames.

Before we concentrate on the Moore–Read state, let us note that observation (iii) confirms the suspicion based on the form of triplet pseudopotential $V_3(\mathcal{R}_3)$ of Fig. 4(c) that (against an earlier assumption²⁶) the Halperin paired state¹² is not an adequate description for systems with subharmonic pseudopotentials at short range. In particular (against our earlier expectation⁴¹ but in agreement with our later numerical results³⁴) such model appears inappropriate for the QE’s in CF-LL₁ at $\nu = \frac{1}{2}$ or $\frac{1}{3}$, corresponding to the FQH states at $\nu = \frac{3}{8}$ and $\frac{4}{11}$. Instead of Halperin’s pairing, grouping of pairs into larger clusters seems to occur for the QE’s, although we are not able to define their correlations more specifically.

Let us now discuss observations (i) and (iii) in more detail. In Fig. 6 we plot $\mathcal{G}_3(3)$ as a function of α for $N = 6$ to 14 (only even values, because the Moore–Read state at $2l = 2N - 3$ is a paired state). For each N , $\mathcal{G}_3(3)$ drops to essentially a zero at exactly $\alpha_0 \approx \frac{1}{2}$, but it increases quickly when α moves away from this critical value. This result is consistent with the calculations of overlaps of the exact ground states of modified Coulomb interaction with the exact Moore–Read trial state.^{25,35}

It is difficult to reliably extrapolate the values of α_0 obtained from Fig. 6 to an infinite (planar) system. However, we notice the following connection with Fig. 4(d) that depends on $2l$ much more regularly. The pair amplitudes $\mathcal{G}_2 = [\mathcal{G}_2(1), \mathcal{G}_2(3), \dots]$ of the $\mathcal{R}_3 < 9$ triplets can be calculated. On a sphere, they slightly depend on $2l$ (on curvature), but the values appropriate for a plane (the $g \rightarrow \infty$ limit) are: $[\frac{3}{4}, \frac{1}{4}]$, $[\frac{9}{16}, \frac{1}{8}, \frac{5}{16}]$, and $[\frac{3}{16}, \frac{5}{8}, \frac{3}{16}]$ for $\mathcal{R}_3 = 3, 5$, and 6 , respectively. Using

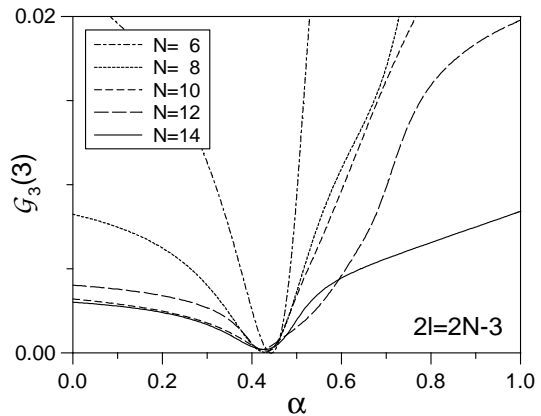


FIG. 6: The $\mathcal{G}_3(3)$ vs. α curve for $\nu = \frac{1}{2}$ shown in Fig. 5(a), magnified and replotted for different particle numbers N .

these values and Eq. (2) one can determine the range α over which $V_3(\mathcal{R}_3)$ is superlinear at short range. The requirement that $\frac{1}{2}[V_3(3) - V_3(5)] > V_3(5) - V_3(6)$ and $V_3(5) - V_3(6) > V_3(6) - V_3(7)$ limits α to a rather narrow window of approximately

$$0.5 < \alpha + \frac{1}{4l} < 0.58. \quad (16)$$

For a reason we do not completely understand (but that is connected with a neglected and complicated behavior of $V_3(\mathcal{R}_3)$ at $\mathcal{R}_3 > 7$), the value of α_0 in finite systems (see Fig. 6) is much closer to the lower limit of Eq. (16). Therefore, we expect that α_0 will follow this lower limit with increasing $2l$, and the value appropriate for a planar system should be even closer to $\frac{1}{2}$ than the finite-size results of Fig. 6. And since $U_{\frac{1}{2}}$ accurately models Coulomb interaction in LL₁, we conclude that the “ $\mathcal{R}_3 > 3$ ” correlations must be an accurate description for experimental $\nu = \frac{5}{2}$ FQH state (even in narrow samples). This conclusion is quite different from an earlier discussion of finite-size numerical wavefunctions^{25,35} which seemed to imply that a $\sim 10\%$ short-range enhancement of the Coulomb pseudopotential calculated for $w = 0$ in LL₁ is needed to reach good overlap with the Moore–Read state.

IV. ENERGY SPECTRA OF SHORT-RANGE THREE-BODY REPULSION

Knowing that what defines the Moore–Read state is that electrons in $\frac{1}{2}$ -filled LL₁ completely avoid the $\mathcal{R}_3 = 3$ triplet state,²⁶ let us discuss the energy spectra of the model short-range three-body repulsion

$$W(\mathcal{R}_3) = \delta_{\mathcal{R}_3,3} \quad (17)$$

which induces precisely this type of correlations. Similar calculations for slightly smaller systems were earlier carried out by Wen²⁷ and by Read and Rezayi.³⁰

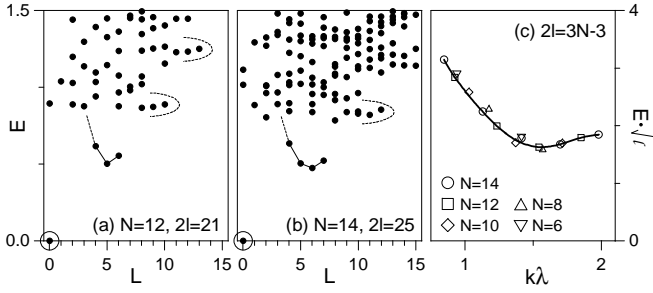


FIG. 7: (a,b) Energy spectra (total interaction energy E vs. total angular momentum L) calculated on a sphere for even numbers of particles N interacting through triplet pseudopotential defined by Eq. (17), at the values of $2l = 2N - 3$ corresponding to the $L = 0$ Moore–Read ground state. (c) Energy dispersion (excitation energy E as a function of wavevector k) for the excited magnetoroton band. λ is the magnetic length. Similar results were first obtained by Read and Rezayi³⁰.

The three-body interaction matrix elements needed for diagonalization in the configuration interaction (CI) basis are connected with the triplet spectrum $V_3(\mathcal{R}_3, \beta_3)$ through expansion parameters $C_B^A \equiv \langle A|B \rangle$ analogous to the pair Clebsch–Gordan coefficients,

$$\begin{aligned} & \langle m_1, m_2, m_3 | V_3 | m_4, m_5, m_6 \rangle \\ &= \sum_{\mathcal{R}_3, \beta_3} C_{m_1, m_2, m_3}^{\mathcal{R}_3, \beta_3*} C_{m_4, m_5, m_6}^{\mathcal{R}_3, \beta_3} V_3(\mathcal{R}_3, \beta_3). \end{aligned} \quad (18)$$

For $V_3 = W$ the above formula reduces to just one term. However, diagonalization of V_3 is still far more difficult than of a (also L -conserving) V_2 because of a larger number of nonzero CI matrix elements (by over 10 times in the systems discussed further in this section).

A. Laughlin-like $\nu = \frac{1}{2}$ incompressible ground state and excited magnetoroton band

In Fig. 7(a) and (b) we present the results for $N = 12$ and 14 and $2l = 2N - 3$. As seen on these two examples, for even values of N and for $2l = 2N - 3$ there is exactly one state in the spectrum with $E = 0$, i.e., with no triplet amplitude at $\mathcal{R}_3 = 3$. In other words, the Hilbert subspace with $\mathcal{R}_3 > 3$ for all triplets contains exactly one state in this case. At $2l < 2N - 3$, all states have amplitude at $\mathcal{R}_3 = 3$, and at $2l > 2N - 3$ there is more than one such state. For odd values of N , no such states occur for at $2l \leq 2N - 3$, and at $2l > 2N - 3$ there are always more than one. This fact makes the Moore–Read yet another beautiful extension of the Laughlin idea for the $\nu = \frac{1}{3}$ state at $2l = 3N - 3$ being the only state in its Hilbert space with no pair amplitude at $\mathcal{R}_2 = 1$. Just as the avoidance of more than one pair state generated the whole $\nu = \frac{1}{3}, \frac{1}{5}, \dots$ sequence, the avoidance of not just pairs, but triplets (or K -body states) gives rise to incompressibility at new values of ν .

The analogy to the Laughlin $\nu = \frac{1}{3}$ state goes beyond the incompressible ground state. The low-energy excitations clearly form a band that resembles the magnetoroton curve.³⁰ In frame (c) we overlay data obtained for different $N = 6$ to 14 and plotted as a function of wavevector k (the charge-neutral excitations carrying $L > 0$ on a sphere move along great circles of radius R , but on a plane they would move along straight lines with $k = L/R$). The continuous character of this band and the minimum at $k \approx 1.5 \lambda^{-1}$ (very close to $k \approx 1.4 \lambda^{-1}$ of the Laughlin $\nu = \frac{1}{3}$ state) are clearly visible.

B. Pairing and Laughlin pair–pair correlations

Before we move on to the spectra at $2l \neq 2N - 3$ in search of the elementary charge excitations of the Moore–Read state, let us recall Halperin’s¹² concept of Laughlin states of $\mathcal{R}_2 = 1$ pairs that we have also used earlier for the half-filling of both LL_1 ¹⁶ and CF-LL_1 .^{34,41} The increase of $\mathcal{G}_2(1)$ compared to a Laughlin-correlated state at the same ν visible in Fig. 3(a) can be thought of as pairing for both $\alpha \sim \frac{1}{2}$ and 1. However, whether the \mathcal{R}_1 pairs will keep far apart from one another by avoiding small values of their relative (pair–pair) angular momentum (what we would consider Laughlin correlations among the pairs) has not been established in neither LL_1 nor CF-LL_1 . Actually, the fact that only for $N = 6, 10, 14, \dots$ (and not for $N = 8$ or 12) do the $L = 0$ ground states occur in CF-LL_1 suggests that Halperin’s idea could not be correct for the interacting QE’s. However, for the half-filled LL_1 , the occurrence of a large value of $\mathcal{G}_2(1)$ and, *at the same time*, the vanishing of $\mathcal{G}_3(3)$ finally offers support for this idea in the Moore–Read state. By effectively acting like a short-range three-body repulsion W , Coulomb repulsion in LL_1 allows grouping electrons into pairs (at ν as large as $\frac{1}{2}$), but it prevents the third electron from getting too close to a pair. As a result, the pairs exist but each pair attains a hard-core that results in Laughlin correlation with all other pairs (or unpaired electrons), and that can be modeled by a fictitious flux attachment in a standard way.

Let us demonstrate how does this picture works for the spectra in Fig. 7. As a result of the appropriate CF transformation,^{16,34,41} N electrons at $2l = (2N + 3) \pm \Delta$ are converted to $N_2 = \frac{1}{2}N$ CF’s about exactly filling their effective CF-LL_0 shell with $2l_0^* = 2(2l - 1) - 7(N_2 - 1) = (N_2 - 1) \pm 2\Delta$, i.e., with the effective degeneracy

$$g_0^* = N_2 \pm 2\Delta. \quad (19)$$

These CF’s correspond to the $\mathcal{R}_2 = 1$ pairs of electrons, and their effective angular momentum l_0^* is obtained from $L_2 = 2l - 1$ by attachment of 7 flux quanta to each pair (4 to account for the pair–pair hard-core due to Pauli exclusion principle, 4 to model pair–pair Laughlin correlations, and 1 in the opposite direction to convert the pairs to fermions). At exactly $2l = 2N - 3$, the N -body (Moore–Read) ground state is equivalent to a full CF-LL_0

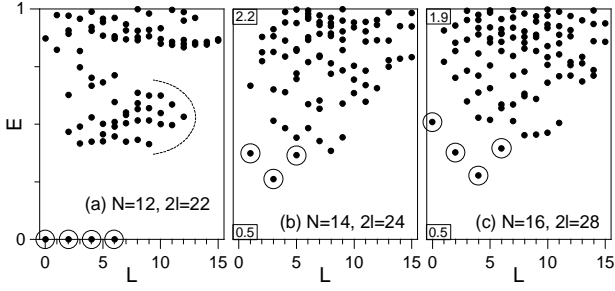


FIG. 8: The same as Fig. 7 but for $2l = 2N - 2$ (a) and $2l = 2N - 4$ (b,c) corresponding to two QH's and two QE's in the Moore-Read state, respectively.

with $l_0^* = \frac{1}{2}(N_2 - 1)$, i.e., to a Laughlin state of N_2 pairs. The magnetoroton band describes QE-QH pair states, with one CF excited from the full CF-LL₀ to the empty CF-LL₁ with $l_1^* = l_0^* + 1 = \frac{1}{2}(N_2 + 1)$. This band extends up to $L = l_0^* + l_1^* = N_2$. Higher states above the magnetoroton band contain additional QE-QH pairs, and the characteristic steps are clearly visible in the energy spectra in Fig. 7 (e.g., at $L = (2l_0^* - 1) + (2l_1^* - 1) = N - 2$ for two QE-QH pairs).

C. Quasiparticles

In Fig. 8 we present sample spectra obtained for even values of N and $2l = (2N - 3) \pm 1$. At $2l = (2N - 3) + 1$, there is always a band^{27,30} of $E = 0$ states at $L = N_2, N_2 - 2, \dots$, corresponding to two QH's in CF-LL₀ of degeneracy $g_0^* = N_2 + 2$. This is shown in frame (a) for $N = 12$. Unlike for Laughlin $\nu = \frac{1}{3}$ state of unpaired electrons, the increase of $2l$ by unity from the value corresponding to a full CF-LL₀ creates not one but two QH's, as predicted by Eq. (19) for our picture of Laughlin-correlated pairs. Note that the same is true for the finite-size Jain $\nu = \frac{2}{5}$ states with two CF LL filled; however, no combination $(N, 2l)$ corresponds to a single QH in a finite-size paired Laughlin state (the condition $g_0^* = N_2 + 1$ leads to a half integral value of g), while for the Jain $\nu = \frac{2}{5}$ state it occurs for even N , at $2l = \frac{1}{2}(5N - 7)$. Similarly as in Fig. 7, the first excited band above the 2QH states contains an additional QE-QH pair, and it extends to $L = (3l_0^* - 3) + l_1^* = N$, exactly as marked in frame (a).

At $2l = (2N - 3) - 1$ no states can have $E = 0$, but the lowest band is expected to contain two QE's in CF-LL₁ of degeneracy $g_1^* = N_2$. Indeed, in spectra (b) and (c) obtained for $N = 14$ and 16 , the low-energy bands at $L = N_2 - 2, N_2 - 4, \dots$ can be found as expected (although they are not as well resolved as the QH bands).

What is the electric charge \mathcal{Q} of the QE's and QH's? Being proportional to the LL degeneracy, it can be obtained from the ratio of g^* and $g = N/\nu$ calculated in the $N \rightarrow \infty$ limit. For a Jain $\nu = n/(2pn + 1)$ state of n completely filled CF-LL's, the degeneracy of each

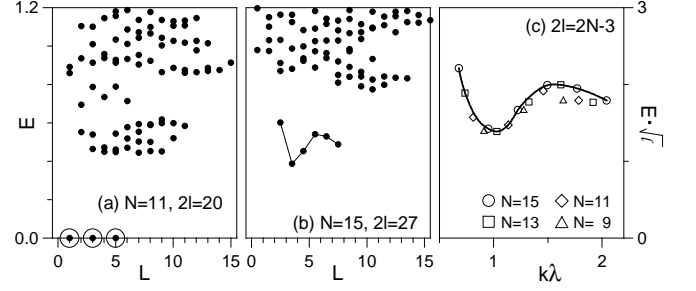


FIG. 9: The same as Figs. 7 and 8 but for odd particle numbers N and for $2l = 2N - 2$ (a) and $2l = 2N - 3$ (b). Energy dispersion (c) is for the pair-breaking band marked in frame (b). λ is the magnetic length.

one is $g^* = N/n$, which leads to the well-known result $\mathcal{Q}/e = g^*/g = (2pn + 1)^{-1}$. For the present case, $g = 2N$, $g^* = N_2 = \frac{1}{2}N$, and the result is precisely what should be expected for a $\nu = \frac{1}{8}$ state of $2e$ -charged boson pairs¹²

$$\mathcal{Q} = e/4. \quad (20)$$

D. Spectra for odd particle numbers

If Halperin's¹² picture could be simply extended to finite $\nu = \frac{1}{2}$ systems with odd electron numbers N , they would contain $N_2 = \frac{1}{2}(N - 1)$ pairs and $N_1 = 1$ unpaired electron, forming a two-component Laughlin-correlated fluid.¹⁶ What actually happens is presented in two sample energy spectra in Fig. 9(a,b).

At $2l = (2N - 3) + 1$ there is a band of $E = 0$ states that indeed correspond to a pair of QH's of the two-component fluid. In the CF picture, each QH has $l_0^* = \frac{1}{2}(N_2 + 1)$ which gives the total $L = N_2, N_2 - 2, \dots$, exactly as obtained for $N = 11$ in frame (a).

At $2l = 2N - 3$ no $E = 0$ states occur, and the numerical results for different N always show a band at $L = \frac{5}{2}, \frac{7}{2}, \dots, \frac{1}{2}N$, that seems to describe dispersion of an excitonic state of a pair of QP's of opposite charge. This becomes more convincing in Fig. 9(c), where the data obtained for different N is plotted together as a function of wavevector k , and a clear magnetoroton-type minimum appears at $k \approx 1.0 \lambda^{-1}$. Remarkably, the values $l = \frac{1}{4}(N \pm 5)$ of the QP angular momenta that would explain the observed range of L do not agree with the prediction of a Laughlin-correlated state with $N_2 = \frac{1}{2}(N - 1)$ and $N_1 = 1$. Nevertheless, knowing their angular momenta is enough to predict the charge $\pm e/4$ of these (not completely identified) QP's.

The reason why this low-energy band cannot be described by a two-component CF model (for any combination of N_1 and N_2 , not just the one with $N_1 = 1$) is that they are not pair-pair or pair-electron, but pair-breaking excitations introduced by Greiter *et al.*²⁶ Such excitations generally occur in paired systems and they are expected to be charge-neutral (despite being fermions)

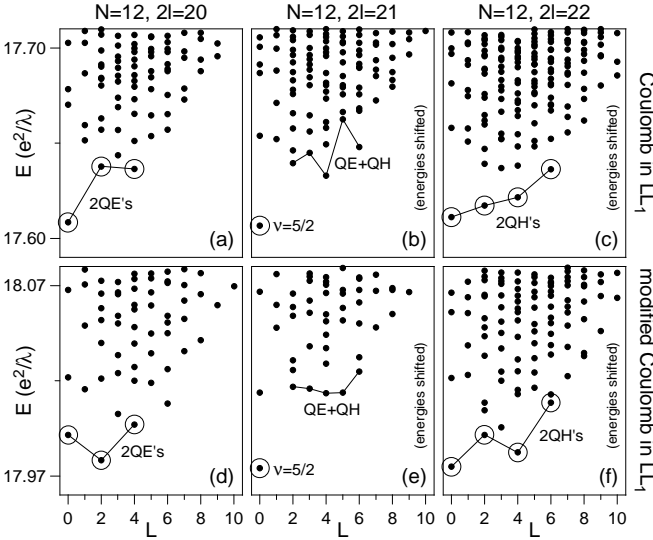


FIG. 10: (a,b,c) Energy spectra similar to those in Figs. 7–9, but obtained for the Coulomb pair pseudopotential of the first excited LL in Fig. 1(b). (d,e,f) Spectra of the same systems obtained for parameter $V_2(1)$ increased by 9%.

which explains their continuous energy dispersion in a magnetic field. Still, the above discussion suggests that it should be possible to decompose them into a pair of more elementary, charged QP's.

V. RELEVANCE TO THE $\nu = \frac{5}{2}$ FQH STATE

Earlier diagonalization studies^{16,25,26,35} using Coulomb pseudopotential in LL_1 showed the $L = 0$ ground states with a gap at $2l = 2N - 3$ but no clear indication of QP excitations identified³⁰ in the spectra of the model three-body repulsion W . As shown in the top frames of Fig. 10 obtained for $N = 12$ electrons, the magnetoroton QE–QH band and of the two-QE bands can indeed hardly be found in these spectra due to mixing with higher states, and only the two-QH bands are well separated.

The problem with the identification of the Coulomb $\nu = \frac{1}{2}$ ground state in LL_1 with the Moore–Read (or any other) trial state is that the former is very sensitive to the relative values of the leading pseudopotential coefficients, while the exact form of $V_2(\mathcal{R}_2)$ depends (at least in principle) on the layer width w in experiments, and on N in finite-size calculations. As to the width-dependence, it turns out that increasing w from zero to realistic experimental values only weakly affects the nearly harmonic behavior of $V_2(\mathcal{R}_2)$ at short range that is essential for the avoidance of the $\mathcal{R}_3 = 3$ triplet state. As a result, the $\nu = \frac{5}{2}$ wavefunction in experimental systems depends much less on the width than, e.g., the excitation gap controlled by the magnitude of V_2 .

On the other hand, the strong dependence of correlations on $\alpha \sim \frac{1}{2}$ in finite systems is clear in Figs. 3(a), 5(a), and 6, and it is in contrast with the behavior at $\alpha \sim 0$

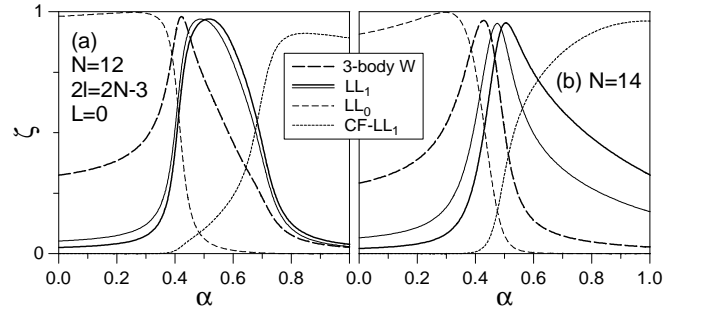


FIG. 11: Squared overlaps ζ of the lowest $L = 0$ eigenstate of pair interaction U_α defined by Eq. (5) calculated on a sphere at $2l = 2N - 3$, with the corresponding eigenstates of three-body repulsion W (Moore–Read state), electron interaction in the lowest and excited LL (the narrow solid line is for layer width $w = 3.5\lambda$), and QE interaction in the Laughlin $\nu = \frac{1}{3}$ state, plotted as a function of α . Frames (a) and (b) correspond to $N = 12$ and 14 particles.

or 1, corresponding to the much less sensitive finite-size FQH states in LL_0 and CF- LL_1 . Remarkably, the gap above the incompressible ground state at $2l = 2N - 3$ persists³⁴ over a wide range of α despite even a large distortion of its wavefunction, while the QP excitations quickly mix with the continuum of higher states when V_2 becomes too sub- or superharmonic at short range.

A major problem with the calculations on a sphere is the size-dependence (16) of the critical value of α at which the avoidance of $\mathcal{R}_3 = 3$ occurs. It is clearly visible in the plots of squared overlaps $\zeta_u(\alpha) = |\langle \phi_\alpha | \psi_u \rangle|^2$ with the eigenstates ϕ_α of U_α , calculated for the corresponding eigenstates ψ_u of various other interactions u : three-body repulsion W and electron and QE pair pseudopotentials V_2 in LL_0 , LL_1 , and CF- LL_1 , respectively. For LL_1 , the overlaps ζ_{LL1} have been calculated for both narrow ($w = 0$) and wide ($w = 3.5\lambda$; e.g., $w = 20$ nm at $B = 20$ T) layers. Note also that the eigenstate or W used in the calculation of overlaps is automatically properly symmetrized (in the original form²⁴ it is not²⁵).

In Fig. 11 we plot the overlaps for the lowest $L = 0$ states at $2l = 2N - 3$. Clearly, the exact Moore–Read eigenstate of W is an excellent ground state of U_α at $\alpha \approx 0.425$. So is the ground state of Coulomb pair interaction in LL_1 , but at a different $\alpha \approx 0.5$. The disagreement between these two values of α does not disappear in wide samples, as inclusion of w even as large as 3.5λ does not noticeably change the Coulomb $\nu = \frac{5}{2}$ ground state. Specifically, the overlaps between the Moore–Read state and the Coulomb $\nu = \frac{5}{2}$ ground state calculated for $N = 14$ are only $|\langle \psi_W | \psi_{CL1} \rangle|^2 = 0.48, 0.58$, and 0.71 for $w/\lambda = 0, 1.75$, and 3.5 , respectively.

The behavior $\zeta_{QE}(\alpha)$ plotted with narrow dotted lines is also noteworthy. The QE–QE interaction at a half-filling can be described by U_1 quite well for $N = 14$ (where the calculations indicate a finite-size $L = 0$ ground state with a gap) and somewhat worse for $N = 10$ (where the ground state is compressible). But even more

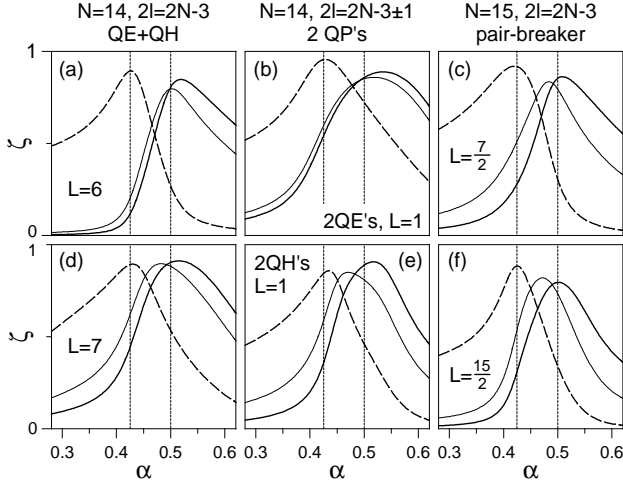


FIG. 12: Similar to Fig. 11 (solid and dashed lines mean the same) but for different low-energy states at $2l = (2N - 3)$ and $(2N - 3) \pm 1$, corresponding to pair QE and QH states and the pair-breaking excitation of the three-body interaction W .

interestingly, the Moore–Read state appears nearly orthogonal to the QE states (the exact value for $N = 14$ is $|\langle \psi_W | \psi_{QE} \rangle|^2 = 0.03$), which we interpret as yet another strong indication against the QE pairing at $\nu = \frac{3}{8}$.

In Fig. 12 we plot similar overlaps calculated for various excitations. Frames (a,d) correspond to a QE–QH pair, (b) to two QE’s, (e) to two QH’s, and (c,f) to the pair-breaking neutral-fermion excitation. We only show the curves for the QE–QH states at $L = 6$ and 7 near the magnetoroton minimum, for two-QE and -QH states at small $L = 1$ (corresponding to large QP–QP separation for which the curves are less dependent on QP–QP interaction effects), and for the pair-breaker at $L = \frac{7}{2}$ near the energy minimum and at a large $L = \frac{15}{2}$. All frames show similar behavior to Fig. 11, only the disagreement between the eigenstates of W and the Coulomb eigenstates is more pronounced. The QP excitations of the three-body repulsion W remarkably well describe actual excitations of a system with a two-body interaction U_α . However, not for the value of α corresponding to the Coulomb interaction in LL_1 (regardless of the layer width). The overlaps between eigenstates of W and the electron eigenstates in LL_1 are even lower than those for the Moore–Read state. The specific values for $N = 14$ and $w = 0$ (and for $w = 3.5\lambda$ in parentheses) are: $|\langle \psi_W | \psi_{C1} \rangle|^2 = 0.03, 0.00, 0.27, 0.19, 0.12, 0.46$ (0.03, 0.02, 0.39, 0.31, 0.20, 0.60) for the $L = 2, 3, \dots, 7$ states of the magnetoroton QE+QH band, 0.47, 0.16, 0.07 (0.52, 0.28, 0.14) for the $L = 1, 3, 5$ states of two QE’s, and 0.39, 0.12, 0.39, 0.27 (0.53, 0.17, 0.64, 0.32) for the $L = 1, 3, 5, 7$ states of two QH’s, respectively. The values for the pair-breaking band for $N = 13$ are: 0.45, 0.19, 0.41, 0.31, 0.34 (0.56, 0.34, 0.44, 0.46, 0.47) for $L = \frac{5}{2}, \frac{7}{2}, \dots, \frac{13}{2}$, respectively. Such small overlaps preclude (indicated) interpretation of excited states in

Fig. 10(a) and (b) in terms of the QE’s of W .

This invokes the question raised in the introduction of whether the Moore–Read trial state and its QP excitations are only an elegant idea, not realized in known even-denominator FQH states (at $\nu = \frac{5}{2}$ or $\frac{3}{8}$). Fortunately, the disagreement appears to be largely artificial. The size-dependence (16) of α_0 can be traced to the size-dependence of the pair amplitudes $\mathcal{G}_2(\mathcal{R}_2)$ of the triplet eigenstates at $\mathcal{R}_3 = 3, 5, 6, \dots$, directly caused by the surface curvature. It is therefore only due to this curvature that (in finite systems on a sphere) $\alpha_0 < \frac{1}{2}$ is different from the value $\alpha = \frac{1}{2}$ appropriate for the Coulomb pseudopotential in LL_1 . This appears to be consistent with larger overlaps calculated for the Moore–Read state in toroidal geometry.²⁵

At $N \rightarrow \infty$, we expect that $\alpha_0 \approx \frac{1}{2}$ in coincidence with the behavior of $V_2(\mathcal{R}_2)$ in the same limit, and that the energy spectra of Coulomb V_2 and model W interactions should become similar. To improve the agreement at $N \leq 14$, for which we were able to calculate the spectra, $V_2(1)$ must be slightly enhanced in accordance with Eq. (16). E.g., for $N = 12$ the near vanishing of $\mathcal{G}_3(3)$ at $2l = 2N - 3$ occurs when $V_2(1)$ is increased by 9% from its Coulomb value, in good agreement with the result of Morf.³⁵ The $N = 12$ electron energy spectra calculated for this interaction with marked features associated with the QP’s are shown in bottom frames of Fig. 10.

The above discussion yields the following statements: (i) Finite-size calculations on a sphere using Coulomb pair interaction do not correctly reproduce correlations of an infinite $\nu = \frac{5}{2}$ state. They use pseudopotentials corresponding to $\alpha \approx \frac{1}{2}$, different from $\alpha_0 < \frac{1}{2}$ leading to the avoidance of $\mathcal{R}_3 = 3$. The $\alpha = \alpha_0 = \frac{1}{2}$ coincidence is probably recovered for $N \rightarrow \infty$ which would mean that the real, infinite systems at $\nu = \frac{5}{2}$ do have the “ $\mathcal{R}_3 > 3$ ” correlations while the correlations in finite systems are different and size-dependent. (ii) In finite systems, correct “ $\mathcal{R}_3 > 3$ ” correlations are recovered if the pair pseudopotential is appropriately enhanced at short range. (iii) Assuming that the $\alpha = \alpha_0 = \frac{1}{2}$ coincidence is restored in infinite systems (or in different, e.g., toroidal geometry), the equivalence of Coulomb and W interactions at half-filling is not limited strictly to the Moore–Read ground state. The $(\pm e/4)$ -charged QP’s and the neutral-fermion pair-breaker identified in the spectra of W accurately describe the low-energy charge excitations in the real (Coulomb) $\nu = \frac{5}{2}$ systems. Although the effective interactions between QP’s may lead to their binding or dressing (just as at $\nu = \frac{1}{3}$ QH’s and “reversed-spin” QE’s bind to form skyrmions), they are simple objects with an elegant interpretation in terms of Laughlin-like three-body correlations.

VI. CONCLUSION

We have studied two- and three-body correlations in partially filled degenerate shells for various interactions

between the particles. Variation of the relative strength of two leading pair pseudopotential coefficients drives the correlations through three distinct regimes. The intermediate regime, corresponding to the nearly harmonic pseudopotential at short range, describes correlations among electrons in LL_1 , particularly in the $\nu = \frac{5}{2}$ FQH state.

In contrast to the correlations between electrons in LL_0 or between Laughlin QE's in CF- LL_1 (whose pseudopotentials are strongly super- and subharmonic at short range, respectively), the intermediate regime is not characterized by a simple avoidance of just one pair eigenstate corresponding to the strongest anharmonic repulsion. Instead, we have shown that near the half-filling the low-energy states for such interactions have simple three-body correlations. In resemblance of Laughlin pair correlations, they consist of the maximum avoidance of the triplet state with the smallest relative angular momentum $\mathcal{R}_3 = 3$, i.e., with the smallest area spanned by the three particles (in analogy to pair correlations, avoidance means here the minimization of a triplet amplitude).

In particular, at exactly half-filling, this corresponds to the fact²⁶ that the Moore–Read ground state is the zero-energy eigenstate of a model short-range three-body repulsion W with the only pseudopotential parameter at $\mathcal{R}_3 = 3$. The Moore–Read ground state is a three-body analog of the Laughlin $\nu = \frac{1}{3}$ state with $\mathcal{R}_2 > 1$. It is separated by a finite excitation gap from a magnetoroton band with a minimum at $k \approx 1.5 \lambda^{-1}$. Its elementary excitations are the $(\pm e/4)$ -charged QP's (that naturally occur for the Halperin¹² state with Laughlin correlations between pairs) and the pair-breaking exci-

tation. The bands of few-QP states near half-filling are well described by a CF picture appropriate for Laughlin pair-pair correlations.

Finally, the problem of numerical calculations on a sphere associated with the surface curvature is addressed. It is found that finite-size models using Coulomb interaction between electrons do not correctly reproduce correlations of the $\nu = \frac{5}{2}$ FQH state due to the distortion of triplet wavefunctions. Especially for the excitations of the $\nu = \frac{5}{2}$ ground state, the overlaps with the Moore–Read-like correlated states are rather small. However, it is argued that the $\nu = \frac{5}{2}$ FQH state observed experimentally in narrow systems is described much better by the Moore–Read trial state than could be expected from the calculation of overlaps in small systems. Consequently, the origin of its incompressibility is precisely the avoidance of the $\mathcal{R}_3 = 3$ triplet state, and its elementary excitations are the $(\pm e/4)$ -charged QP's (although more complex excitations, such as bound QP states, might be the lower-energy charge carriers in realistic systems).

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